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Minimum-Time Control Characteristics of Flexible Structures

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The time-optimal control for rest-to-rest maneuvers of flexible structures is considered. We formulate the general time-optimal control problem for single-axis flexible structures, and analytical results are given for the number of control switches for the one-bending-mode case, with and without damping. When there is no damping, it is shown that the time-optimal control generally has three switches and is an odd function of time about the second switch except in certain isolated cases where there is only one switch. With damping, it is shown that there is always more than one switch. A numerical method is presented for solving the time-optimal control for general linear systems, and solutions are presented for flexible structures with several flexible modes, revealing interesting trends of the time-optimal control switch times as the maneuver sizes and the frequencies and damping ratios of the flexible modes are varied.

I. Introduction

IN many applications, such as manipulators, disk-drive heads, or pointing systems, sophisticated control algorithms are required to make optimal use of the maximum torque available for rapid maneuvers.[1–4] In recent years, for speed and fuel efficiency purposes, bulky rigid structures have rapidly been replaced by lightweight flexible structures.[5,6] The requirement of time minimization results in a bang-bang control that can be implemented in many applications using current on-off actuation technology. Solving for the time-optimal control for flexible structures, however, has posed a challenging problem. The time-optimal control for general maneuvers and general flexible structures is still an open problem. Solving for the time-optimal control for rest-to-rest slewing maneuvers of flexible structures has been an active area of research, but only limited solutions have been obtained.[7–11]

In this paper, we present results on the characteristics of the time-optimal control for rest-to-rest maneuvers of undamped and damped flexible structures. For the one-bending-mode case without damping, we show that the number of switches in the time-optimal control is usually three except in isolated cases where there is only one switch. Since all real systems have some damping, we also address the problem of time-optimal control of flexible structures with damping. When there is damping in the one-bending-mode case, the number of switches is proven to always be greater than one.

Since the time-optimal control of multimodal models of flexible structures is difficult to solve analytically, we outline a numerical technique for solving the open-loop time-optimal controls for general linear systems. We then present results of using this numerical method for solving the time-optimal controls of flexible structures with various modal frequencies and damping ratios, verifying properties derived analytically as well as revealing several additional characteristics. Finally, we also discuss the errors that result when the damping is ignored in the design of the time-optimal control.

The paper is organized as follows. In Sec. II, we formulate the problem of time-optimal control for rest-to-rest maneuvers of flexible structures. In Sec. III, we present analytical results outlining some of the properties of the time-optimal control switch times for the one-bending-mode case, with and without damping. A numerical method for solving open-loop time-optimal controls for general linear systems is outlined in Sec. IV, and numerical results and simulations demonstrating some interesting characteristics of the time-optimal control solutions for flexible structures are presented. Finally, concluding remarks are given in Sec. V.

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II. Problem Formulation

A single-input model of a flexible structure with damping is

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

where $A = \mathrm{blockdiag}[A_0, A_1, \ldots, A_n]$ and

$$b = [b_0 \quad 0 \quad b_1 \quad \cdot \quad \cdot \quad \cdot \quad 0 \quad b_n]^T$$

The term A_0 represents the nonflexible (rigid body) dynamics of the system

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b_0 = [0 \quad 1]^T$$

and

$$A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix}, \qquad i = 1, 2, \ldots, n$$

represent the flexible dynamics of the system where $\omega_1 < \cdots < \omega_n$ are the structural frequencies, and ζ_i, $i = 1, \ldots, n$, are the damping ratios of the flexible modes.

We consider the rest-to-rest time-optimal control problem, where the objective is to choose the scalar control function $u(t)$ satisfying

$$|u(t)| \le U_0 \tag{2}$$

so that the motion is transferred from the initial state

$$x(0) = (-L, 0, 0, \ldots, 0, 0)^T \tag{3}$$

to the final state

$$x(t_f) = \mathbf{0} \tag{4}$$

where the total time t_f is minimum.

Since the system is linear, marginally stable, controllable, and normal,[12–14] existence and uniqueness of the optimal solution are guaranteed. The time-optimal control is bang bang with a finite number of switches. Pontryagin's maximum principle gives the following sufficient and necessary conditions for the optimal control, which are denoted by $(\cdot)^*$,

$$\dot{p}^*(t) = -A^T p^*(t), \qquad t \in [0, t_f^*] \tag{5}$$

$$u^*(t) = -U_0 \mathrm{sgn}[b^T p^*(t)], \qquad t \in [0, t_f^*] \tag{6}$$

$$\mathcal{H}(t_f^*) = 0 \tag{7}$$

where $\mathcal{H}(t)$ is the Hamiltonian

$$\mathcal{H}(t) = 1 + \boldsymbol{p}^T(t)[\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)] \tag{8}$$

and

$$\boldsymbol{p}(t) = [p_0(t) \quad q_0(t) \quad p_1(t) \quad q_1(t) \quad \cdots \quad p_n(t) \quad q_n(t)]^T \tag{9}$$

is the costate vector, and the signum function is defined as

$$\mathrm{sgn}(\xi) = \begin{cases} +1, & \xi > 0 \\ 0, & \xi = 0 \\ -1, & \xi < 0 \end{cases} \tag{10}$$

The boundary conditions (3) and (4) impose the following constraints on the switching and maneuver times:

$$2t_1 - 2t_2 + 2t_3 - \cdots + 2(-1)^{k-1}t_k + (-1)^k t_f = 0 \tag{11}$$

$$-2(t_1)^2 + 2(t_2)^2 - \cdots + 2(-1)^{k-1}(t_{k-1})^2 + 2(-1)^k(t_k)^2 + (-1)^{k+1}t_f^2 = 2L/(\alpha U_0) \tag{12}$$

$$1 - 2e^{\omega_i c_i t_1} + \cdots + 2(-1)^k e^{\omega_i c_i t_k} + (-1)^{k+1} e^{\omega_i c_i t_f} = 0 \tag{13}$$

$$1 - 2e^{\omega_i d_i t_1} + \cdots + 2(-1)^k e^{\omega_i d_i t_k} + (-1)^{k+1} e^{\omega_i d_i t_f} = 0 \tag{14}$$

where

$$c_i = \zeta_i + \sqrt{\zeta_i^2 - 1} \qquad i = 1, 2, \ldots, n \tag{15}$$

$$d_i = \zeta_i - \sqrt{\zeta_i^2 - 1} \qquad i = 1, 2, \ldots, n \tag{16}$$

and where α is the initial sign of the control $u(0)$, and k is the number of switches in the time-optimal control.

Condition (5) implies that

$$\boldsymbol{p}(t) = e^{-A^T t}\boldsymbol{p}(0) \tag{17}$$

The switching function of Eq. (6) satisfies

$$\boldsymbol{p}^T(t)\boldsymbol{b} = 0 \text{ at the switch times } t_1, t_2, \ldots, t_k \tag{18}$$

Computing the matrix exponential in Eq. (17) and substituting in Eq. (18) give k equations. Together with the transversality condition (7), we obtain $k + 1$ conditions that can be written as a matrix equation:

$$\begin{bmatrix} -t_f & 1 & b_1 f_{1f} & b_1 g_{1f} & \ldots & b_n f_{nf} & b_n g_{nf} \\ -t_k & 1 & b_1 f_{1k} & b_1 g_{1k} & \ldots & b_n f_{nk} & b_n g_{nk} \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ -t_2 & 1 & b_1 f_{12} & b_1 g_{12} & \ldots & b_n f_{n2} & b_n g_{n2} \\ -t_1 & 1 & b_1 f_{11} & b_1 g_{11} & \ldots & b_n f_{n1} & b_n g_{n1} \end{bmatrix} \begin{bmatrix} p_0(0) \\ q_0(0) \\ \cdot \\ \cdot \\ \cdot \\ p_n(0) \\ q_n(0) \end{bmatrix} = \begin{bmatrix} (-1)^{k+1}\alpha/U_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where

$$f_{ij} = f(\omega_i, \zeta_i, t_j) = \frac{e^{\omega_i d_i t_j} - e^{\omega_i c_i t_j}}{2\omega_i\sqrt{\zeta_i^2 - 1}} \tag{20}$$

$$g_{ij} = g(\omega_i, \zeta_i, t_j) = \frac{c_i e^{\omega_i c_i t_j} - d_i e^{\omega_i d_i t_j}}{2\sqrt{\zeta_i^2 - 1}} \tag{21}$$

and c_i and d_i are defined in Eqs. (15) and (16). The index $i = 1, 2, \ldots, n$ in Eqs. (20) and (21) refers to the ith flexible mode of the system, and the index $j = 1, 2, \ldots, k, f$ corresponds to a switching or maneuver time. Equation (17), and hence Eq. (19), can also be expressed as a function of the costate vector (9) evaluated at a time t_p other than 0 by redefining the time argument in the costate transition matrix to be relative to t_p:

$$\boldsymbol{p}(t) = e^{-A^T(t-t_p)}\boldsymbol{p}(t_p) \tag{22}$$

Equations (11–14) and (19) represent a system of $(2n + 2) + (k+1)$ nonlinear algebraic equations in $(2n+2)+(k+2)$ unknowns which are the k switching times $t_i, i = 1, 2, \ldots, k$; the maneuver time t_f; the initial conditions of the costates $p_i(0)$ and $q_i(0)$, $i = 0, 1, \ldots, n$; and the initial sign of the control α.

Since Eqs. (11–14) and (19) have been derived from Eqs. (1) and (2–7), they are necessary conditions for optimality. To fully satisfy Eq. (6), the optimal switching function $\boldsymbol{p}^T(t)\boldsymbol{b}$ must vanish only at the switching times $t_i, i = 1, 2, \ldots, k$, on the interval $t \in [0, t_f]$. That is,

$$\boldsymbol{p}^T(t)\boldsymbol{b} \neq 0 \qquad \text{for} \qquad t \neq t_i, \qquad i = 1, 2, \ldots, k \tag{23}$$

Although Eqs. (11–14) and (19) may have multiple solutions, only one solution will also satisfy Eq. (23).[8,12]

Solving Eqs. (11–14) and (19) analytically for the time-optimal controls for general flexible structures is a formidable task. In the next section, we present analytical results for the one-bending-mode case ($n = 1$); in Sec. IV, we outline a numerical method and solve for the time-optimal control for flexible structures with multiple flexible modes, and we observe several general characteristics of the behavior of the time-optimal control of flexible structures.

III. Analytical Results

In this section, we prove some analytical results for the characteristics of the time-optimal control of the flexible system of Eq. (1). The minimum time problem for flexible structures with no damping has been addressed in several technical papers.[3,7,8] In Ref. 3, the structure of the control is shown to have an odd number of switches and to be an odd function of time about the middle switch. However, no statement is made on how many switches is optimal. In Refs. 3 and 7, statements are made that in most cases the optimal control for a flexible structure with n bending modes has $2n + 1$ switches. In Sec. III.A, we prove that when $n = 1$, there are never more than three switches in the time-optimal control.

Since there is always some damping in real flexible structures, the case with damping is addressed in Sec. III.B. The equations for this case are much more complex than the case for no damping. We show that in the one-bending-mode case there is always more than one switch in the time-optimal control for rest-to-rest maneuvers.

A. Case with No Damping

The following two results were proven in Ref. 3 for $\zeta = 0$.

1) The optimal control is an odd function of time about the middle switch:

$$u^*(t) = -u^*\left(-t + t_f^*\right), \qquad t \in \left[0, t_f^*/2\right] \tag{24}$$

2) The optimal costate vector (9) is such that at midmaneuver:

$$\boldsymbol{p}\left(t_f^*/2\right) = \left[p_0\left(t_f^*/2\right) \quad 0 \quad p_1\left(t_f^*/2\right) \quad 0 \quad \cdots \quad p_n\left(t_f^*/2\right) \quad 0\right]^T \tag{25}$$

These properties are used to prove the following theorem.

Theorem 1. The time-optimal control for rest-to-rest maneuvers for the system (1) with one flexible mode ($n = 1$) with no damping ($\zeta = 0$) is bang bang with at most three reversals and with the initial sign of the control $\alpha = \mathrm{sgn}(L)$.

Proof. The result follows if Eqs. (5–7) are satisfied for three reversals with $\alpha = \mathrm{sgn}(L)$. Using Eq. (24), we only need to solve for two times $t_a = t_f/2 - t_1$ and $t_f/2$, where switching occurs at $t_1 = t_f/2 - t_a$, $t_2 = t_f/2$, and $t_3 = t_f/2 + t_a$. For the undamped one-bending-mode case, Eqs. (11–14) that are required to satisfy the boundary conditions (3) and (4) reduce to

$$(t_f/2)^2 - 2t_a^2 = L/(\alpha U_0) = |L/U_0| \tag{26}$$

$$\cos(\omega t_f/2) - 2\cos(\omega t_a) + 1 = 0 \tag{27}$$

The property of Eq. (25) suggests that we rewrite Eq. (19) using Eq. (22) in Eq. (18) with $t_p = t_f/2$ to require the switching function to vanish at the switching times and to satisfy Eq. (7). This gives

$$p_0(t_f/2)\, t_f/2 + \bar{b}_1 p_1(t_f/2)\sin(\omega t_f/2) = -\alpha/U_0 \tag{28}$$

$$p_0(t_f/2)\, t_a + \bar{b}_1 p_1(t_f/2)\sin(\omega t_a) = 0 \tag{29}$$

where $\bar{b}_1 = b_1/\omega$. Equations (26–29) are necessary conditions for optimality, and they are also sufficient provided

$$\boldsymbol{p}^T(t)\boldsymbol{b} \neq 0, \quad \text{for } t \in [0, t_f] \text{ except at } \{t_f/2 - t_a, t_f/2, t_f/2 + t_a\} \tag{30}$$

where $\boldsymbol{p}(t) \in \mathcal{R}^4$ (for one flexible mode) is defined by Eqs. (5) and (9) and satisfies Eq. (25) at midmaneuver.

Thus, we need to show that Eqs. (28–30) are always satisfied using the minimum t_a and t_f found from Eqs. (26) and (27). From Eq. (26), we can write $t_f/2$ in terms of t_a:

$$t_f/2 = \sqrt{2t_a^2 + N} \tag{31}$$

where $N = |L/U_0| > 0$. Substituting this into Eq. (27) gives

$$\cos\left(\omega\sqrt{2t_a^2 + N}\right) - 2\cos(\omega t_a) + 1 = 0 \tag{32}$$

Since $t_f/2$ can be written in terms of t_a, we can write all of the optimality conditions in terms of t_a.

After solving for $\boldsymbol{p}(t_f/2)$ from Eqs. (28), (29), and (31), we can obtain the following expression for the switching function:

$$\boldsymbol{p}^T(\bar{t})\boldsymbol{b} = \frac{\bar{t}\,\sin(\omega t_a) - t_a\sin(\omega\bar{t})}{\alpha U_0\left[\sqrt{2t_a^2 + N}\,\sin(\omega t_a) - t_a\sin\left(\omega\sqrt{2t_a^2 + N}\right)\right]} = \frac{n_{sw}(\bar{t})}{d_{sw}(\bar{t})} \tag{33}$$

where $\bar{t} = t - t_f/2$. If this switching function satisfies Eq. (30), i.e., if $\boldsymbol{p}^T(\bar{t})\boldsymbol{b} = 0$ when $\bar{t} = -t_a$, 0, or t_a, then t_a and t_f are optimal. Since the switching function is also an odd function about the midtime point, it suffices to look only at $\bar{t} \in [0, t_f/2]$.

For any $N > 0$, there is exactly one $t_a \in (0, \pi/(2\omega)]$ that satisfies Eq. (32), and this t_a yields the smallest t_f in Eq. (31). To show that Eq. (30) holds, it is convenient to consider two cases: $t_a = 0$ and $t_a \in (0, \pi/(2\omega)]$.

Part 1. For $t_a = 0$, there is only one switch, and the solution is equivalent to the rigid body solution. The trivial case is $N = 0$; if the step size is 0, then no control is needed! For $t_a = 0$ and $N > 0$, the switching function $\boldsymbol{p}^T(\bar{t})\boldsymbol{b}$ can be shown to be 0 only at $\bar{t} = 0$ using l'hôpital's rule.

Part 2. For $t_a \in (0, \pi/(2\omega)]$, it is obvious that $n_{sw}(\bar{t}) = 0$ at $\bar{t} = 0$. For $\bar{t} > 0$, $n_{sw}(\bar{t}) = 0$ implies that $\sin(\omega t_a)/(\omega t_a) = \sin(\omega\bar{t})/(\omega\bar{t})$ and the only $\bar{t}$ in the interval $[0, t_a]$ satisfying this equation is $\bar{t} = t_a$.

Now we show that $d_{sw} \neq 0$ for $t_a \in (0, \pi/(2\omega)]$. From Eq. (32),

$$\sin\left(\omega\sqrt{2t_a^2 + N}\right) = 2\sqrt{\cos(\omega t_a) - \cos^2(\omega t_a)}$$

so that

$$d_{sw} = (\sigma\beta - \delta\gamma)/U_0$$

where

$$\sigma = \sqrt{2t_a^2 + N}, \qquad \delta = \sqrt{2}\, t_a$$

$$\beta = \sqrt{1 - \cos^2(\omega t_a)}, \qquad \gamma = \sqrt{2\cos(\omega t_a) - 2\cos^2(\omega t_a)}$$

For $t_a \in (0, \pi/(2\omega)]$, $\sigma > \delta$ and $\beta > \gamma$, so that $d_{sw} > 0$.

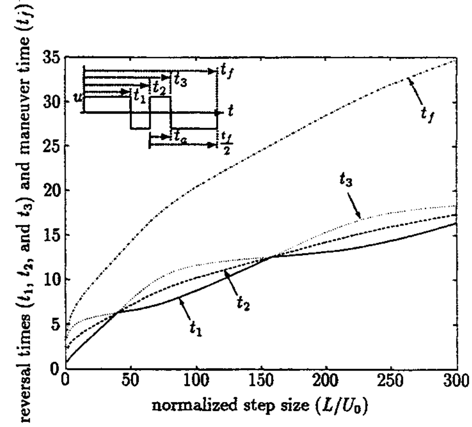

Fig. 1 Absolute optimal reversal and maneuver times vs step size for a rigid body with one undamped flexible mode (ω = 1, ζ = 0, and b_1 = 1).

Thus, Eqs. (26–30) are fulfilled, and the time-optimal solution for Eqs. (1–4) with $n = 1$ and $\zeta = 0$ has either one switch or three switches. If $N = (2m\pi)^2/\omega^2$ for some $m = 0, 1, 2, \ldots$, then $t_a = 0$; the time-optimal solution only has one switch and is equivalent to the rigid-body solution. Otherwise, the time-optimal solution has three switches. □

A plot of the absolute optimal reversal and slewing times as a function of step size is shown in Fig. 1 for $\omega = 1$. For a fixed $N = |L/U_0|$ in Eq. (32), t_a generally decreases as ω increases; but depending on N, t_a can occasionally increase as ω increases due to the periodic nature of the cosine function. With this behavior of t_a substituted in Eq. (31), we can conclude that for a constant step size L and constant actuator limit U_0, the optimal slewing time t_f generally decreases as ω increases; however, it does not decrease monotonically.

B. Case with Damping

When there is damping in the flexible mode of the structure, the control is no longer an odd function about the midtime point, and solving for the time-optimal control is far more complex than for the case without damping. Here we establish a lower bound on the number of switches for the one-bending-mode case. Although the time-optimal control of an undamped one-bending-mode model of a flexible structure has only one switch for certain size maneuvers, the time-optimal control for damped flexible structures never has only one switch.

Theorem 2. The time-optimal control for rest-to-rest maneuvers for the system (1) with one flexible mode ($n = 1$) with damping ($\zeta > 0$) is bang bang with more than one switch.

Proof. We will break the proof into three parts: 1) $0 < \zeta < 1$, 2) $\zeta = 1$, and 3) $\zeta > 1$.

Part 1. $0 < \zeta < 1$. If one switch in the time-optimal control could achieve a step maneuver of size L, then from Eqs. (11–14), we must have

$$t_1 = \sqrt{N} \tag{34}$$

$$t_f = 2t_1 \tag{35}$$

$$1 - 2e^{\omega\zeta\sqrt{N}}\cos[\omega\sqrt{N(1-\zeta^2)}] + e^{2\omega\zeta\sqrt{N}}\cos[2\omega\sqrt{N(1-\zeta^2)}] = 0 \tag{36}$$

$$-2e^{\omega\zeta\sqrt{N}}\sin[\omega\sqrt{N(1-\zeta^2)}] + e^{2\omega\zeta\sqrt{N}}\sin[2\omega\sqrt{N(1-\zeta^2)}] = 0 \tag{37}$$

where $N = L/(\alpha U_0)$. [From Eq. (34), since $t_1 > 0$, α must be $\mathrm{sgn}(L)$.] Let $q = \omega\sqrt{[N(1-\zeta^2)]}$ and $v = \omega\zeta\sqrt{N}$. Then Eq. (37) can be rewritten as

$$-2e^{v}\sin(q) + 2e^{2v}\sin(q)\cos(q) = 0$$

or

$$\sin(q)[1 - e^{v}\cos(q)] = 0$$

This holds if and only if

$$\sin(q) = 0 \quad \text{or} \quad 1 - e^{v}\cos(q) = 0 \tag{38}$$

Similarly, Eq. (36) can be rewritten as

$$1 - 2e^{v}\cos(q) + 2e^{2v}\cos^2(q) - e^{2v} = 0$$

or

$$2e^{v}\cos(q)[1 - e^{v}\cos(q)] + (e^{2v} - 1) = 0 \tag{39}$$

If $[1 - e^{v}\cos(q)] = 0$, then $e^{2v} = e^{2\omega\zeta\sqrt{N}} = 1$, which implies that either $\omega = 0$ or $\zeta = 0$ (for nontrivial maneuvers N). But we are assuming that $\omega > 0$ and $\zeta > 0$. Thus $[1 - e^{v}\cos(q)] \neq 0$, and from Eq. (38), $\sin(q) = 0$, which implies that $\cos(q) = \pm 1$. For $\cos(q) = +1$, Eq. (39) becomes

$$2e^{v}(1 - e^{v}) + e^{2v} - 1 = 0$$

or

$$2e^{v} - e^{2v} - 1 = 0 \tag{40}$$

For $\cos(q) = -1$, Eq. (39) becomes

$$-2e^{v}(1 + e^{v}) + e^{2v} - 1 = 0$$

or

$$2e^{v} + e^{2v} + 1 = 0 \tag{41}$$

Let $r = e^{v} \geq 0$. Then from Eq. (40), $r^2 - 2r + 1 = 0 \Rightarrow r = 1$; and from Eq. (41), $r^2 + 2r + 1 = 0 \Rightarrow r = -1$. The latter case is not possible since $r \geq 0$. For $r = e^{v} = e^{\omega\zeta\sqrt{N}}$, this implies that either $\omega = 0$ or $\zeta = 0$, again contradicting our assumption that $\omega > 0$ and $\zeta > 0$. Thus, there is no solution with $0 < \zeta < 1$ that has only one control switch.

The proofs for parts 2 and 3 ($\zeta \geq 1$) are similar and hence are not detailed. □

A plot of the optimal reversal and slewing times as a function of step size is shown in Fig. 2 for $\zeta = 0.01$. The plot is similar to the case of no damping except for step sizes near where there is only one switch in the no damping case. In these regions, there are always three switches; however, two of them are so close together that for all practical purposes the control can be considered to have only one switch. A study of how the optimal maneuver time varies with damping shows that it increases as damping increases.[11]

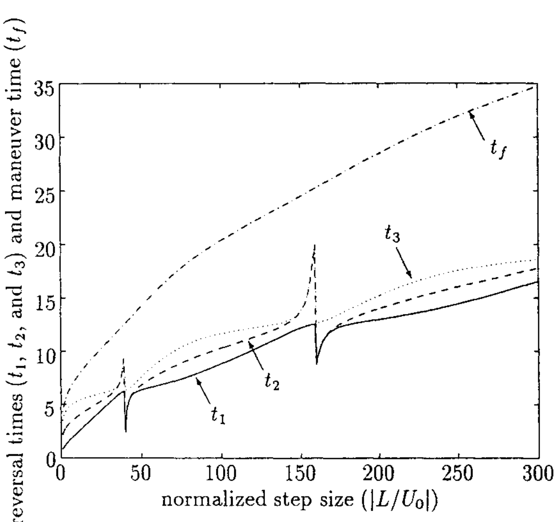

Fig. 2 Absolute optimal reversal and maneuver times vs step size for a rigid body with one lightly damped flexible mode ($\omega = 1$, $\zeta = 0.01$, and $b_1 = 1$).

IV. Numerical Method

In this section, we review the numerical method of Ref. 15 for solving the open-loop time-optimal controls for given maneuvers for general linear systems. The method formulates the problem as a set of constrained least-squares problems and uses a standard linear least-squares program for numerically solving the problem.

For a continuous-time system (1), the discrete-time model (using a zero-order hold) is

$$\boldsymbol{x}_{k+1} = \boldsymbol{\Phi}\boldsymbol{x}_k + \boldsymbol{\Gamma}u_k \tag{42}$$

where k is the time index and[16]

$$\boldsymbol{\Phi} = e^{AT} \tag{43}$$

$$\boldsymbol{\Gamma} = \int_0^T e^{A\xi}\,\mathrm{d}\xi\,\boldsymbol{b} \tag{44}$$

where T is the sampling time (so that kT is the time at index k). The state at index k can be written as

$$\boldsymbol{x}_k = \boldsymbol{\Phi}^k \boldsymbol{x}_0 + \mathcal{C}\mathcal{U} \tag{45}$$

where $\mathcal{C}$ is the matrix

$$\mathcal{C} \triangleq [\boldsymbol{\Phi}^{k-1}\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}^{k-2}\boldsymbol{\Gamma} \quad \ldots \quad \boldsymbol{\Phi}\boldsymbol{\Gamma} \quad \boldsymbol{\Gamma}] \tag{46}$$

and $\mathcal{U}$ is a vector of controls up to the time index $k - 1$. That is,

$$\mathcal{U}^T = [u_0 \quad u_1 \quad \ldots \quad u_{k-2} \quad u_{k-1}] \tag{47}$$

The initial and final states $\boldsymbol{x}(0)$ and $\boldsymbol{x}(t_f)$ are given by Eqs. (3) and (4). Formulating the problem as a set of constrained least-squares problems, the problem is to find the smallest k such that there exists a $\mathcal{U}$ (where $|u_i| \leq U_0$ for $i = 0, 1, \ldots, k-1$) such that the 2-norm of

$$\mathcal{L}(\mathcal{U}) = \boldsymbol{\Phi}^k \boldsymbol{x}_0 + \mathcal{C}\mathcal{U} \tag{48}$$

is zero. The problem of finding

$$\mathcal{J} = \min_{|u_i| \leq U_0,\, i = 0,1,\ldots,k-1} (\mathcal{L}^T\mathcal{L})^{\frac{1}{2}}$$

can be solved using a linear least-squares solver such as LSSOL.[17] If $\mathcal{J} > 0$, then kT is smaller than the minimum time. If $\mathcal{J} = 0$, then kT is greater than or equal to the minimum time. The index k can be varied until the smallest k such that $\mathcal{J} = 0$ is found. If the sampling time T is small, then kT will be a close approximation to the minimum time for the maneuver in the continuous-time case; the control sequence $\mathcal{U}$ will be a close approximation to the time-optimal control in the continuous-time case as well. The true time optimality of numerically obtained results can be verified by checking that the necessary and sufficient conditions (5–7) of Pontryagin's maximum principle are satisfied.

A. Characteristics of the Time-Optimal Control

Using the numerical technique described earlier, we solved for the time-optimal controls for the flexible system (1) with $n = 1$, 2, or 3 flexible modes for a wide range of ω_i and ζ_i as well as maneuver sizes L. The results for some specific cases are shown in Fig. 3, where there are three flexible modes with $\omega_1 = 25$, $\omega_2 = 50$, and $\omega_3 = 100$. The position response curves were obtained by simulating the system of Eq. (1) open loop with u being the time-optimal control obtained using the preceding numerical method. The plots are for step maneuvers of size $L = 1$, and the parameters are $U_0 = 1$, and $b_i = 1$, $i = 1, 2, 3$. A sampling time of $T = 0.0001$ was used in the numerical method.

For the upper plot in the figure, $\zeta_i = 0.3$ for all flexible modes. In the middle plot, $\zeta_i = 0.01$ for all flexible modes. Finally, for comparison, the bottom plot represents the system $A = A_0$ with no flexible modes. We see that the time-optimal control when there are three flexible modes is similar to the time-optimal control with no flexible modes. When there are three flexible modes, there are

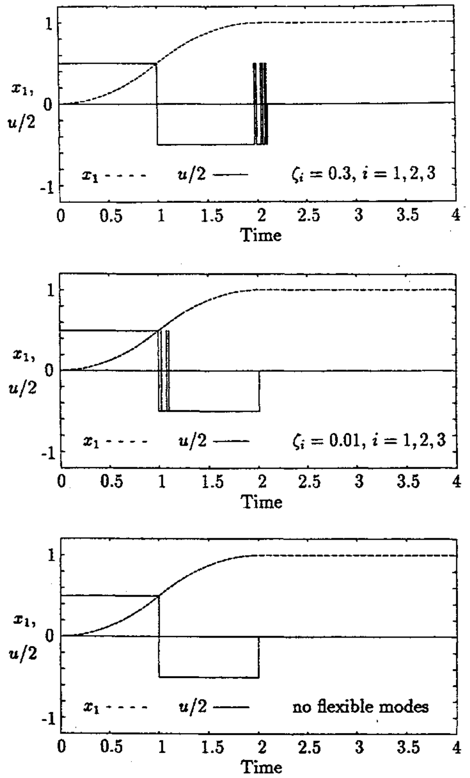

Fig. 3 Time-optimal position and control histories for a step maneuver for a rigid body with three flexible modes (upper two plots: $\omega_1 = 25$, $\omega_2 = 50$, and $\omega_3 = 100$) and for a rigid body with no flexible modes (bottom plot).

six additional control switches to take out energy from the flexible modes. With the higher damping of 0.3 on the flexible modes (uppermost plot), the six additional switches occur at the end of the maneuver. With a much lighter damping of 0.01 on each of the flexible modes (center plot), however, the extra switches occur near midmaneuver.

From solving the time-optimal controls for numerous cases, we found that for larger damping ratios ($\zeta_i > 0.25$) the extra control switches occur at the end of the maneuver, whereas for lightly damped systems ($\zeta_i < 0.1$), the additional switches occur near mid maneuver. (It is worth noting that when ζ is growing, and therefore the dissipative term is becoming dominant, the solution resembles the time-optimal solution of the heat equation where the switches are known to accumulate near the final time.[18]) This behavior is similar to that observed in Sec. III for a one-bending-mode model of a flexible structure. We also found that each additional flexible mode usually causes two additional switches in the time-optimal control, resembling similar observations in Sec. III and Ref. 3. However, as seen in the next figure, this is not always the case.

Figure 4 shows how the switch and maneuver times vary as a function of damping. For Fig. 4, there is one flexible mode whose frequency is $\omega_1 = 10$, and the switch and maneuver times are for unit step maneuvers $L = 1$ with $U_0 = 1$ and $b_1 = 1$. The $\times$ symbols in the figure denote where the numerical method was applied. We see the characteristic behavior of the switch times being near the end of the maneuver for $\zeta \geq 0.24$ and well before the end of the maneuver for $\zeta \leq 0.16$. In these ranges, there are three switches (two additional switches for the flexible mode beyond the one switch for the rigid-body mode). For $0.18 \leq \zeta \leq 0.22$, however, there are five switches, with two switches near the end of the maneuver and two switches near the rigid-body switch near the middle of the maneuver. This property of an additional two switches for moderate damping values of approximately $0.1 \leq \zeta \leq 0.25$ (the actual range varies slightly) was observed for several modal frequencies ω_1 and maneuver sizes L.

For the one-bending-mode case, numerical studies also show that, for $\zeta \geq 0.2$, $t_f - t_3$ and $t_3 - t_2$ become constants for large values of $|L/U_0|$ (see Fig. 5). If $\zeta \gg 1$ (where the system is no longer a

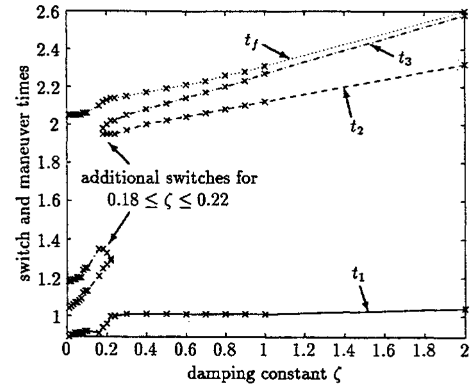

Fig. 4 Variation of switch and maneuver times for a rigid body with one flexible mode as a function of the damping ratio ($\omega_1 = 10$).

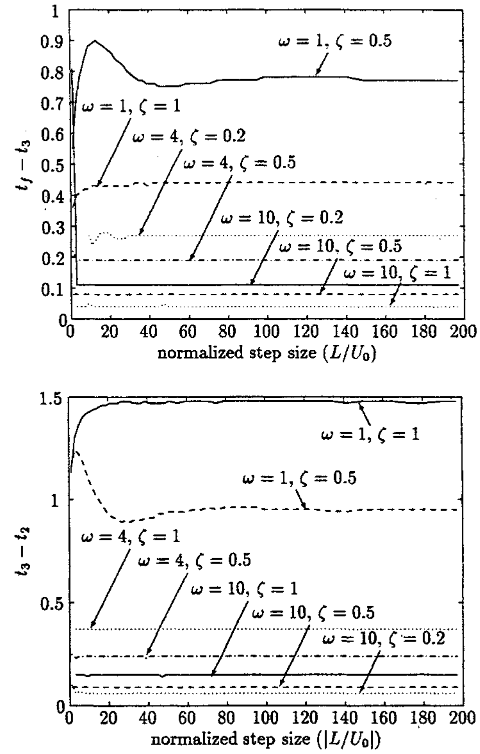

Fig. 5 For $\zeta \geq 0.2$, the differences $t_f - t_3$ and $t_3 - t_2$ converge to constants as $|L/U_0|$ increases.

flexible structure), $t_f - t_3$ approaches the constant 0 regardless of step size (see Fig. 6); that is, the time-optimal control essentially only has two switches for large ζ. (In Fig. 6, for cases where there are five switches rather than three, the plot shows the difference between t_f and t_5.)

These numerical studies verify the analytical results of Sec. III and further reveal additional properties of the time-optimal control switch times for flexible structures. Although analytical solutions are extremely difficult to obtain for multimodal models of flexible structures, numerically solving for the time-optimal controls has proven to be relatively easy. Further, since it appears the switch times have certain properties and are fairly well behaved, numerical solutions for a sparse range of frequencies and damping ratios can give good insights into the behavior of the time-optimal controls over wider ranges of frequencies and damping values.

Given these properties, we can develop control schemes that are independent of maneuver size. An area of future work is that of investigating a feedforward/feedback method that consists of constructing a switching curve based on the rigid-body mode and then superposing a feedforward command consisting of the additional switches. For highly damped systems, the additional switches would be injected near the end of the maneuver when the rigid-body

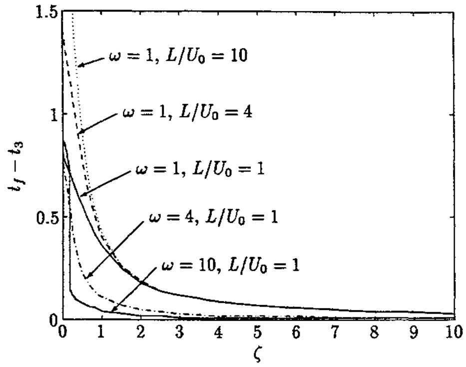

Fig. 6 Difference between the maneuver time t_f and the switch time t_3 becomes negligible as ζ gets very large.

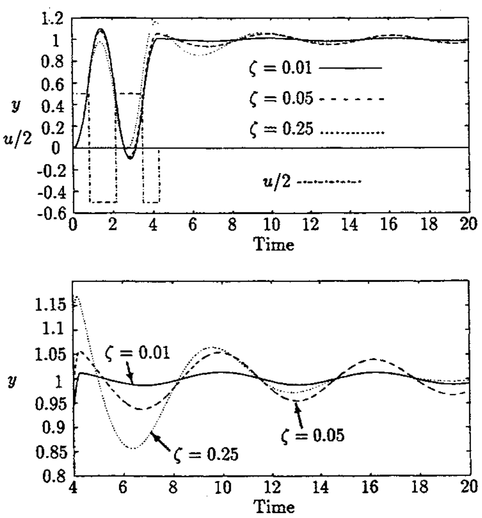

Fig. 7 Residual output position errors from applying the time-optimal control for an undamped system to a system with damping (one flexible mode, $\omega_1 = 1$).

position becomes close to the desired position. The feedback position error would be used to determine when the feedforward control switches are applied. Similarly, for lightly damped systems, the additional switches would be applied near midmaneuver based upon when the rigid-body position becomes close to the switching curve. Medium damped systems would have additional switches both when the rigid-body position is near the switching curve as well as near the desired position. It would also be of interest to determine whether other feedforward control methods for flexible structures[19–23] have similar behaviors that are essentially independent of maneuver size. (The time-optimal controls solved here are feedforward open-loop commands.) If so, similar feedforward/feedback controllers could be developed with these approaches.

Finally, we present some results showing the errors that remain when the damping is ignored. We assume a system output y of the form

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t) + du(t) \tag{49}$$

where $\boldsymbol{c} = [c_0\ 0\ c_1\ 0\ \cdots\ c_n\ 0]$. Each ratio c_i/c_0, $i = 1, 2, \ldots, n$, determines the relative magnitude of the ith flexible modal position to that of the rigid-body position. Note that although the time-optimal control is independent of $\boldsymbol{c}$ (and d), the errors caused by ignoring the damping do depend on $\boldsymbol{c}$ as well as the structural frequencies ω_i.

The results in Fig. 7 show the residual errors that occur when the damping is ignored for a system with one damped flexible mode, i.e., the time-optimal control applied is that for an undamped system. The frequency of the flexible mode in all cases is $\omega_1 = 1$, and $c_0 = c_1 = 1$ and $d = 0$. The upper plot shows the entire maneuver, whereas the lower plot shows the errors after the control has been completed. We see that the larger the damping, the larger the initial error after the control is completed. However, the larger the damping, the faster the system naturally dampens out the error, whereas lightly damped systems will continue to oscillate for a much longer time. In general, as the ratios c_i/c_0 decrease and as the frequencies ω_i of the flexible modes increase, the residual errors will decrease and ignoring the damping in the modeling becomes less detrimental in the control performance; this is as expected since the effect of the flexible modes becomes less important under these circumstances.

V. Conclusions

Flexible structures are becoming more and more prevalent in mechanical systems, and these structures inherently have some damping. It is generally very difficult to analytically solve for the time-optimal controls for slewing these flexible structures because the equations are extremely complex and intractable.

We have analytically demonstrated a few properties of the time-optimal control for a one-bending-mode model (with and without damping) of a flexible structure. The time-optimal control for rest-to-rest maneuvers of an undamped one-bending-mode model of a flexible structure is bang bang with at most three switches and is symmetric about the second switch. In isolated cases when the maneuver distance is a particular multiple of the inverse square of the flexible modal frequency, there is only one switch. For the damped one-bending-mode case, we have proven that there is always more than one switch.

For general higher order flexible structures, we have used a numerical method to characterize the behavior of the time-optimal control switch times. Several interesting properties of the time-optimal control have been revealed that are primarily dependent upon damping while being fairly independent of maneuver sizes.

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